Flow Optimization in Transport Networks

Michal Kutil

Department of Control Engineering,
Czech Technical University in Prague,
Technicka 2, 166 27 Prague 6, Czech Republic

kutilm@fel.cvut.cz

Abstract. *This article focuses on the urban traffic control, based on the model by difference state equations. Model is described by the number of vehicles in the queue and mainly by the mean value of waiting time which describes the queue dynamics. To make the appropriate non-linear model and to identify its parameters we use real data measured during one day in Prague. The objective of our work is to balance the vehicle waiting time of different streets in one intersection. For that purpose we desing two controllers (Linear Quadratic Controller and Non-linear Model Predictive Controller) and we simulate behavior of such controlled system assuming real incomming flows.*

1. Introduction

This article focuses on the urban traffic control based on the model by difference state equations, frequently used in the classic control theory [1, 15], one can easily decompose the urban traffic infrastructure into microregions describing particular streets and intersections. To make the appropriate non-linear model and to identify its parameters we use real data measured during one day in Prague. The objective of our work is to balance the vehicle waiting time of different streets in one intersection. For that purpose we design two controllers (Linear Quadratic Controller and Non-linear Model Predictive Controller) and we simulate behavior of such controlled system assuming real incoming flows.

The growth of urban traffic requires efficient control methods. The studies of *Intelligent transportation systems* (ITS) date back to the 1960s. Since then a lot of work has been done (see for example [16]) on road traffic control, freeway traffic control, route guidance and driver information. An isolated intersection control, belonging to the road traffic control problems, is usually based either on a *Fixed-time* strategy or on a *Traffic-response* strategy.

The Fixed-time strategy (e.g. a TRANSYT tool [17]), where the light control phases (i.e. duration of green or red light) are scheduled offline, is optimal only in the case of the undersaturated intersections. The light control phases are derived from the historical data measured on a given intersection. There are several light control phases for each intersection, depending on the given time of the day (for example, the morning one is different from the afternoon one).

The Traffic-response strategy is based on on-line measurements of current state of traffic (e.g. a SCOTT tool [9]). A *Store-and-forward* strategy [5, 6] is one of the Traffic-response strategies based on rigorous mathematical model. The main idea when using store-and-forward models for road traffic control is to introduce a model simplification that enables the mathematical description of the traffic flow process without use of discrete variables. This simplifies the description of the system state space and opens a possibility to use polynomial optimisation and control algorithms.

Our approach builds on the Store-and-forward strategy. We use the real data from detectors placed approximately 300 meters in front of the intersection. In order to derive the incoming flow we process these data via Kalman filter [8], since this approach enables to estimate the queues of the length superior to 300 meters. In addition to the classical Store-and-forward strategies [6] our model incorporates the vehicle waiting time which is a crucial input parameter for the controllers designed in this paper.

This article is inspired by [7], where non-linear difference state equations are used to model and control the web server traffic.

Other modeling approaches, with different precision and complexity, are based on the queuing theory [10, 18] and Petri nets [3].

This paper is organized as follows. Section 2 describes the extended queue model. The microregion model description is given in Section 3 and its control is given in Section 4. The paper summary and future work are described in Section 5.

2. Extended Queue Model

Classical traffic control strategies uses n *number of vehicles in the queue* measured in *unit vehicles* $[uv]$, as an input to the control law with objective to minimize this value. If we want to increase the quality of traffic control from the driver point of view, then we have to add a new objective, a waiting time. The waiting time is a time spend by the vehicle in the queue.

The state vector which describes the queue model is written in the form

$$\mathbf{x_c} = (x_1, x_2, \ldots, x_i)^T. \tag{1}$$

Where x_i denotes the number of vehicles which are waiting in the queue i time unit. This model we denote as *complete queue model*. Disadvantage of this model description is the state equation complexity and mainly unbounded size of the vector $\mathbf{x_c}$. That is why we reduce it to an approximation model.

The approximation model incorporates two state variables. The first one is the number of vehicles in the queue n and the second one is E $[s]$, the *mean value of waiting times*. The mean value of waiting times E is computed as S, the sum of waiting times over all vehicles, divided by n, the number of vehicles. State vector is written in the form $\mathbf{x} = (n, E)^T$. This approximation model will be denoted as an *extended queue model*.

2.1. Geometrical Interpretation of the Extended Queue Model Evolution

To this approach we will derive difference state equations for the extended queue model evolution in the discrete time. Extended queue model evolution is dependent on a vehicles flow and the length of the time unit. Vehicle flow is a volume of vehicles which transfer the queue over the time unit. There is an incoming flow w $[uv \cdot h^{-1}]$ given by the vehicles arriving to the queue and outgoing flow q $[uv \cdot h^{-1}]$ given by the vehicles leaving the queue.

Geometrical interpretation of the model can be depicted by the right-angled triangle as illustrated in the Figure 1 by the bold line. The horizontal leg of the triangle represents n, the number of vehicles in the queue. The vertical triangle leg represents waiting time of the first vehicle in the queue and it is equivalent to the double mean value of waiting times $(2E)$. The area of this right-angled triangle S represents the sum of waiting times over all vehicles. The relationship between E, n and S is

$$E = \frac{S}{n}. \tag{2}$$

To this section we consider stable incoming flow w and outgoing flow q. Model evolution from discrete time k to time $k+1$ can be divided to the tree steps:

1. Outgoing vehicles flow q corresponds to the removal of the polygon A from the main triangle.
2. All vehicles which stay in the queue have to increase their waiting time. In the Figure 1 it is represented by the rectangle C.
3. Incoming vehicles flow w is represented by the new triangle D. This triangle is added to the area.

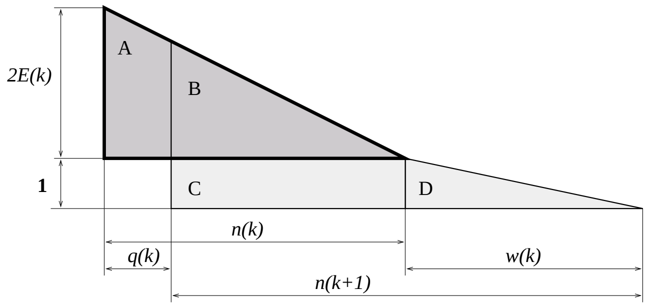


Fig. 1. Extended queue model evolution

The new area is equivalent to the S, the sum of waiting times over all vehicles in time $k+1$. This area $S(k+1)$ can be calculated as the sum of polygons B, C and D areas (S_B, S_C, S_D)

$$S(k+1) = \underbrace{\frac{E(k)(n(k)-q(k))^2}{n(k)}}_{S_B} + \underbrace{n(k)-q(k)}_{S_C} + \underbrace{\frac{w(k)}{2}}_{S_D}. \tag{3}$$

From depicted geometrical interpretation and equations (2) and (3) we can determine the following discrete state equations

$$n(k+1) = n(k) - q(k) + w(k), \tag{4}$$

$$E(k+1) = \frac{\frac{E(k)(n(k)-q(k))^2}{n(k)} + n(k) - q(k) + \frac{w(k)}{2}}{n(k)-q(k)+w(k)}. \tag{5}$$

State equations are valid only for $n(k) > 0$ and $n(k) > q(k) - w(k)$ conditions, it means that there must be some vehicles in the queue, otherwise $E(k+1)$ must be equal to 0.

2.2. Extended Queue Model Evaluation

Presented extended queue model is an approximation of the complete queue model which is described by the state vector (1). The complete queue model has complex mathematical description, but its implementation is possible for bounded number of vehicles. We implemented this queue model in order to evaluate the extended queue model.

For evaluation we used the data from the real traffic region. We used these data as an input to both models. Mean value of waiting time is response measured on the models. The result is shown on the Figure 2. Difference between models is a minimal error of the extended queue model. This error is caused by the non stable incoming flow, which is not available in the real input data. But we can say that the estimated model is sufficient for our future study.

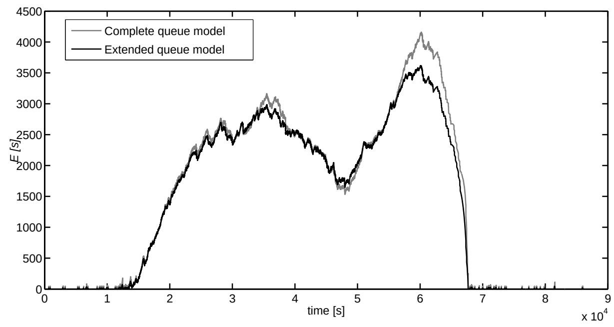


Fig. 2. Queue model evaluation

2.3. Extended Queue Model Equilibrium

Equilibrium is a system state (equilibrium point) in which all state variables are balanced, that is $\mathbf{x}(k) = \mathbf{x}(k+1)$. The equilibrium point is used for linearization issue and further for the control synthesis described in Section 4.

Equilibrium point for our state vector have to fulfill the following conditions

$$n(k) = n(k+1), \quad (6)$$
$$E(k) = E(k+1). \quad (7)$$

Solution of these equations implies the following equilibrium point conditions

$$q^{\circ}(k) = w^{\circ}(k), \quad (8)$$
$$2E^{\circ}(k) = \frac{n^{\circ}(k)}{q^{\circ}(k)}. \quad (9)$$

The circle mark means that the value of given variable is the value in the equilibrium. Condition (8) means that the incoming flow w must be equal to the outgoing flow q. It involves constant number of vehicles in the queue. Second condition (9) implies that mean value of waiting times is proportional to the queue length and inversely proportional to the vehicle flow.

The Condition (9) between E and n in the equilibrium is well known as the Little's law [13]. The Little's theorem interpreted to our terminology says: "The average number of vehicles in a stable queue (over some time interval) is equal to their average incoming flow, multiplied by their average time in the queue."

3. Model of Microregion

The queue model described above will be used to construct a *model of microregion* (see Figure 3). The microregion consists of two streets and one intersection. The street is a place where the queued of vehicles can be found. Intersection is a place where the vehicles from sever queues share the same resource. Outgoing flow q for each queue is controlled by the semaphore at the intersection.

Microregion model can be described as

$$\mathbf{x_M}(k+1) = \mathbf{F}\left(\mathbf{x_M}(k), \mathbf{q}(k), \mathbf{w}(k)\right), \quad (10)$$

where $\mathbf{x_M}(k) = (\mathbf{x_1}(k), \mathbf{x_2}(k))^T$ is a state vector comprising two extended queue models $\mathbf{x_1}(k)$ and $\mathbf{x_2}(k)$ described in the Section 2. The state vector can be decompose into n and E, therefore

$$\mathbf{x_M}(k) = (n_1(k), E_1(k), n_2(k), E_2(k))^T. \quad (11)$$

$\mathbf{F}$ is a non-linear function given by equations (4) and (5). Vector $\mathbf{q}(k) = (q_1(k), q_2(k))^T$ represents outgoing flow for the queues and vector $\mathbf{w}(k) = (w_1(k), w_2(k))^T$ represents incoming flow. Subscript denotes the queue number.

3.1. Linear model

Linear model is constructed via linearization of function $\mathbf{F}$ (10) in the equilibrium point (Section 2.3). Equilibrium point O was selected as an average point in the real traffic situation, this point is described as

Variable	Queue1	Queue2
$q^{\circ} = w^{\circ}\ [uv \cdot h^{-1}]$	100	150
$n^{\circ}\ [uv]$	20	50
$E^{\circ} = \frac{n^{\circ}}{2q^{\circ}}\ [s]$ (9)	360	600

Linearized microregion model is given by the following equation

$$\mathbf{x_M}(k+1) = \mathbf{A}\mathbf{x_M}(k) + \mathbf{B}\mathbf{q}(k) + \mathbf{B_w}\mathbf{w}(k). \quad (12)$$

Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B_w}$ are computed as a solution of a Jacobian matrices from the function $\mathbf{F}$ in the equilibrium point O.

$$\begin{aligned}\mathbf{A} &= J_F(n_1(k), E_1(k), n_2(k), E_2(k)) \\ &= \begin{bmatrix} \frac{\partial n_1}{\partial n_1} & \cdots & \frac{\partial n_1}{\partial E_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial E_2}{\partial n_1} & \cdots & \frac{\partial E_2}{\partial E_2} \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.05 & 0.997 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.02 & 0.998 \end{bmatrix}_{/O},\end{aligned}$$

$$\mathbf{B} = J_F(q_1(k), q_2(k)) = \begin{bmatrix} -1 & 0 \\ -17 & 0 \\ 0 & -1 \\ 0 & -11 \end{bmatrix}_{/O},$$

$$\mathbf{B_w} = J_F(w_1(k), w_2(k)) = \begin{bmatrix} 1 & 0 \\ -17 & 0 \\ 0 & 1 \\ 0 & -11 \end{bmatrix}_{/O}.$$

4. Control of Microregion

The goal of the control is to find an optimal schedule for the intersection lights. In this section two controllers from the modern control theory will be used and compared.

An incoming flow of vehicles arriving to the intersection, has to be separated into several *phases*. The phase separation, designed by the traffic engineers, determines a direction of vehicles driving through the intersection. The repetitive sequence of the phases is a *control period*. The phases have fixed order in a control period and our goal is to find their optimal timing.

The microregion mentioned above includes the two control phases. Each phase allows vehicles to flow only from one street, see Figure 3. Our control algorithms consider a constant sum of the phase time intervals, i.e. constant control period T. In this section the control period will be set to 90 seconds. Time, when the first phase passes to the second one, will be denoted as *switching time* t_{sw}. The switching time can be used to derive the control law for the model (10) as follows:

$$\mathbf{q}(k)_{/t_{sw}} = \begin{cases} (q_{max1}, 0)^T & \text{if } k \in \langle iT,\ iT + t_{sw}), \\ (0, q_{max2})^T & \text{if } k \in \langle iT + t_{sw},\ (i+1)T), \end{cases} \tag{13}$$

where $q_{max\,j}$ is a maximal feasible outgoing flow from the queue j and $i = 0, 1, 2, 3, \dots$ is an index of the control period.

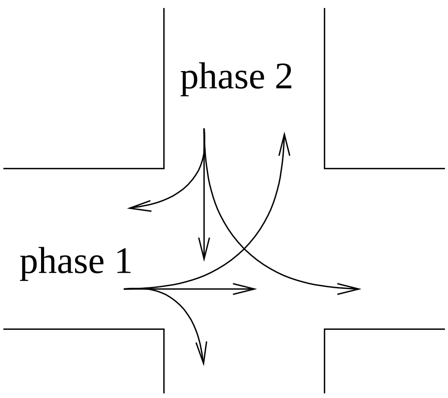


Fig. 3. Street phase

4.1. Linear Quadratic Controller

In this subsection the Linear Quadratic (LQ) controller [11] will be used for traffic intersection control. The controller objective is to minimize the vehicles waiting time. It means that all vehicles incoming into the queue at one moment from different will wait equivalent time. This objective can be writen as

$$J(k) = (E_1(k) - E_2(k))^2. \tag{14}$$

With respect to (11) the objective can be rewritten as

$$J(k) = \mathbf{x_M}(k)^T \mathbf{Q} \mathbf{x_M}(k) \tag{15}$$

where the matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The LQ control law which minimizes the objective over one day time horizon $\sum J(k)$ can be written as $\mathbf{q}'(k) = (q'_1(k), q'_2(k))^T = \kappa \cdot \mathbf{x_M}(k)$. Where κ is a solution of the Riccati equation for the system described on (12). Therefore we derive by simple matrix-vector computations:

$$\kappa = \begin{bmatrix} -0.001 & -0.020 & 0.000 & -0.012 \\ 0.001 & 0.016 & -0.001 & -0.062 \end{bmatrix} \tag{16}$$

This control law produces an unbounded result $\mathbf{q}'(k)$, which can't be directly applied to the microregion control. From this result we have to compute switching time t_{sw} as

$$t_{sw}(k) = T \frac{q'_1(k)}{q'_1(k) + q'_2(k)}.$$

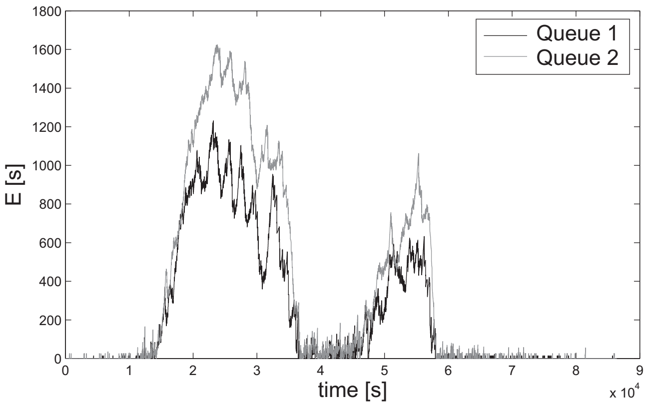
Form the switching time and equation (13) we derive a final control law $\mathbf{q}(k)$. The control law is computed during the control period start and it is hold for the whole period.

The LQ controller was applied to the traffic intersection model control (10). System response to the real signal is shown on the Figure 4(a) with significant difference between waiting time of a Queue1 a Queue2, caused by linearization of the model in equilibrium point.

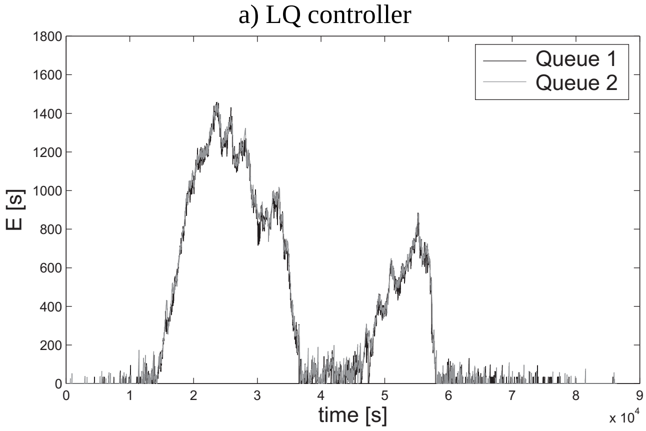
4.2. Non-Linear Model Predictive Controller

The Second controller applied to the microregion control is a Non-Linear Model Predictive (NMPC) controller [4, 14]. The same minimization function $J(k) = (E_1(k) - E_2(k))^2$ was used as an objective. For NMPC controllers we must establish control and predictive horizon. The predictive horizon is a time interval which the controller uses for an optimal actuation search in its model. The control horizon is a time for which the optimal actuation is applied. The control horizon must be shorter or equal to the predictive horizon. Both horizons were set to the 90 seconds, which is equal to the control period T.

The NMPC obviously finds an optimal switching time t_{sw}, by the convex optimalization methods [2]. But our optimization problem isn't convex. That is why we find to optimal t_{sw} over all state space by the exhaustive search. The traffic model response to actuation from the NMPC is shown on the Figure 4(b).



a) LQ controller



b) MPC controller

Fig. 4. Controller application response

The NMPC allows to tune an optimal control signal by the changes of different controller parameters. For example we can extend the controller by known future incoming flow. In practice we can measure this traffic on previous intersection (as shown by [12]) and this traffic we include into the next intersection controller. Predictive controller thus can prepare much better actuation. For example the sum of minimization function value $\sum J(k)$, over all duration of experiment, is equal $19,6 \cdot 10^7$. When we use a known future incoming traffic then the sum is $12,3 \cdot 10^7$.

The function $\sum J(k)$ curve shape for different controllers is depicted on the Figure 5. We can see that the NMPC returns much better results than LQ controller.

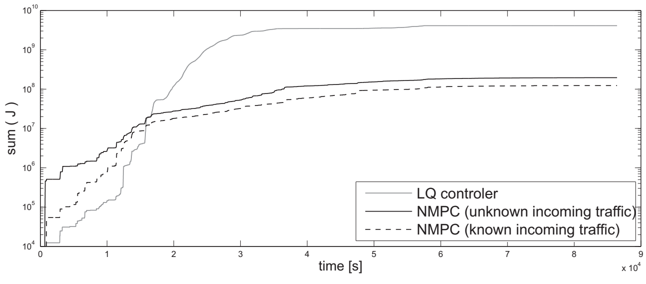


Fig. 5. Evolution of accumulated objective function

On the other hand the NMPC has big time complexity. If we resize the predictive horizon from 90 to 180 seconds then the computation time grows from less than one minute to a five minutes. The horizon resizing to the 270 seconds implies an impossible computing problem in the one day time interval.

5. Summary and Future Work

In this paper, a new model for the traffic queue modeling is presented. The extended queue model is described by the number of vehicles in the queue and mainly by the mean value of waiting time which describes the queue dynamics. The model is based on non-linear difference (i.e. discrete-time) state equations. These equations describe the model behavior in both, the equilibrium point and non-equilibrium states. We have shown that the equilibrium point is in accord with the Little's law.

Further we have used the extended queue model to derive the parameters of the controllers for the traffic microregion with two queues. Two controllers were applied to the microregion control. Firstly we have shown the Linear Quadratic controller with a necessary linearization of the queue equations in the equilibrium point. Secondly the Non-Linear Predictive controller was proposed. The advantages and disadvantages of those controllers were discussed.

Currently we work on the model incorporating several intersections. In our future work we would like to incorporate additional practical constraints to the problem (i.e. switching time constraints imposed by the juridical system or supervisory system performing hierarchical optimisation).

In order to model the traffic with higher precision (i.e. incorporating logarithmic stream model capturing the output flow as non-monotonic function of the car density) we develop a model based on continuous Petri Nets. Further we want to compare both models and evaluate the corresponding control techniques.

Acknowledgements

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